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Finite Lattices under Element Deletion: Structural Properties and Classification

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Abstract

A structure theory over one elementary operation on a finite lattice (namely, the deletion of one element) is theoretically described in this paper. An element x of a finite lattice L is deletable, relative to a structural property that L has, if the poset $L - x$ (which is L with x removed) is also a lattice with the same property. Before discussing the specific theory of deletable elements, dismantlable lattices, the adjunct-sum decomposition, the nullity of a poset, RC-lattices, the reducibility number, and the order dimension of RC-lattices, we establish, from scratch, the order-theoretic tools a operator theory of these concepts requires — distributivity, modularity, semimodularity, and the representation of a finite lattice by its join-irreducible elements. A few formulas are given: the graph-theoretic nullity formula, the edge-count formula for the adjunct sum, the inequality $2 \leq r \leq 2k$ for the number of reducible elements by nullity and the closed-form formula for the number of nullity one lattices (Pawar and Waphare 2003), and the dimension bound of Bhavale (2025). Two of these are hand-checked in full on explicit small lattices. Throughout, the emphasis is on the logical connections between these results, and not on further computation, since it is seen that the reducibility framework is powerful enough to resolve a special case of the Frankl's union-closed sets conjecture, and that the nullity inequality is a consequence of the decomposition into an adjunct-of-chains.

Keywords: finite lattice; irreducible element; deletable element; dismantlable lattice; adjunct sum; nullity; RC-lattice; order dimension.

1. Introduction

At its most elementary level, a finite lattice is a finite partially ordered set L such that for every two elements a, b of L there exists an element a ∨ b of L, as well as an element a ∧ b of L. It is an apparently humble closure condition with extraordinarily rich consequences: finite lattices represent closure systems, congruence structures of algebras, subgroup and subspace lattices, the combinatorics of set systems (and many others). The present paper is concerned with the following question, which is deceptively simple, and can be asked of any finite lattice L: which x of L can be deleted, in the sense that L − x is still a lattice of the same kind as L? The systematic questioning of the answer, which goes back to the 1970's, turns out to organise a large portion of the structure theory and enumeration theory of finite lattices, and even to have an impact on an unsolved problem in combinatorics.

The basic fact that is the foundation of the definition of a dismantlable lattice is that an element in a finite lattice whose only element immediately below it is at most one element, and also whose only element immediately above it is at most one element, is always deletable without destroying the lattice property (due essentially to Rival). Kelly and Rival then provided a purely order-theoretic, forbidden-substructure characterisation of dismantlability: a finite lattice is dismantlable if and only if no retract of it is a crown. Bordalo and Monjardet extended the basic concept of doubly irreducible elements to all deletable elements with respect to any structural property of lattices, and asked for each of the classical varieties of lattices, which are deletable and which are reducible in this sense. Indian scientists, especially at the University of Pune, studied this question in detail and found that, in addition to answering this question for more classes of lattices, they were able to construct a complete structure theorem (an adjunct-sum decomposition of Thakare, Pawar and Waphare) which gives an explicit description of every dismantlable lattice, and used this decomposition to define the nullity of a lattice, the class of RC-lattices, the reducibility number, and — most recently — the order dimension of RC-lattices.

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The paper is structured in a manner from theory to application. The required order-theoretic background is given in sections 2 and 3, where the well-known characterizations of distributivity, modularity and semimodularity are recalled and proved, the proofs being short and illuminating, and the representation of a finite lattice by its join-irreducible elements is recalled. Deletable elements and reducibility are formally introduced in Section 4. The dismantlability and adjunct-sum theory and its elaboration by means of nullity and RC-lattices are developed in sections 5 and 6; here, and nowhere else, a few real formulas are formulated and, in two instances, proved by hand. The reducibility number, the order dimension of RC-lattices and the union-closed sets application are treated in sections 7 to 9. Section 10 discusses the current boundary of what is known and concludes.

2. Order-Theoretic Foundations

We first remember the classical hierarchy of lattice varieties in which the question of deletability is asked: the answer is essentially dependent on the lattice variety under consideration. A lattice L is distributive if for all a, b, c ∈ L, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (and by duality); L is modular if for all a, b, c ∈ L, $a \leq c$ implies that $a \vee (b \wedge c) = (a \vee b) \wedge c$; and L is (upper) semimodular if whenever a and b are both covers of $a \wedge b$, $a \vee b$ covers both a and b. All distributive lattices are modular, and all modular lattices are semimodular, but neither the converse is true.

Theorem 2.1. (Dedekind). A lattice L is modular if and only if it has no lattice of the form $N_5 = \{0, a, b, c, 1\}$ with $0 < a < b < 1$, $0 < c < 1$ and c incomparable with a and b.

The implication in the other direction is immediate: If L is modular and contained an N_5 sublattice on {0, a, b, c, 1} as described, then $a \leq b$ would imply $a \vee (c \wedge b) = (a \vee c) \wedge b$ by modularity; but $c \wedge b = 0$ (since c, b are incomparable in N_5 and 0 is the only common lower bound) so the left side is $a \vee 0 = a$, whereas $a \vee c = 1$ (since a, c have only 1 as common upper bound in N_5) so the right side is $1 \wedge b = b$, giving a = b, a contradiction of a < b. The converse — that the lack of N_5 implies modularity — goes in the opposite direction, and is shown by contraposition: if L is not modular, then an explicit non-modularity of the modular law is exhibited which yields three elements of L whose join and meet together with them form a sublattice order-isomorphic to N_5. The details of this direction are omitted here, as it is standard and not part of the deletable-element theory itself, which is the subject of this paper, and which may be found in Birkhoff (1979) or Grätzer (2011).

Theorem 2.2. (Birkhoff). A lattice L is distributive if and only if it does not contain any sublattice isomorphic to N_5 and any sublattice isomorphic to the diamond $M_3 = \{0, x, y, z, 1\}$ where x, y, z are atoms and coatoms of each other.

The reason for stating these two forbidden-sublattice characterisations is that they will be used in Section 4 to explain in a concrete way what a deletable element looks like in a modular or distributive lattice: deleting an element must never create an N_5 or M_3 configuration where none existed before, and these latter are exactly those controlled by the characterisation theorems of Bordalo and Monjardet.

The second basic fact is a representation theorem which states that a finite lattice is completely determined, as a poset, by its join-irreducibles, together with the order relation between them. From the last talk, remember that an element j in a finite lattice L is join-irreducible if j > 0 and if j is not a join (sum) $b \vee c$ of two elements b and c with b, c < j; by the covering characterisation, this is equivalent to the property that j has a unique lower cover.

Theorem 2.3. (Finite representation theorem). Each element a of a finite lattice L is the join of all of the join-irreducible elements below it, i.e. $a = \bigvee\{ j \in J(L) : j \leq a \}$, where J(L) is the set of join-irreducible elements of L.

Proof. If a is join-irreducible then the statement is trivial, as $a = \bigvee\{a\}$. If not, then go on by induction on the height of a in L. Assume a is not join-irreducible, then there exist b, c < a such

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that a = b ∨ c. By the induction hypothesis, b = ⋁{ j ∈ J(L) : j ≤ b } and c = ⋁{ j ∈ J(L) : j ≤ c }, so a = b ∨ c = ⋁ ({ j ≤ b } ∪ { j ≤ c }), and every join-irreducible element in this union lies below a, since b, c ≤ a. On the other hand, each join-irreducible j ≤ a is involved in at most reaching a from below, and so the join of all of them is ≤ a, which together with the previous inequality yields equality.
The reason that Theorem 2.3 is so important is that it implies the join-irreducible (dually, meet-irreducible) elements of a finite lattice contain all of the combinatorial information of the lattice, and this is the underlying principle that make deletable-element theory important: an element that is neither join- nor meet-irreducible is, in a precise sense, redundant information, expressible in terms of simpler information already available in the lattice.
The weakest of the three properties, semimodularity is most naturally described not as a forbidden sublattice but by a numerical inequality on the height function, which is the point of view taken later in this paper, when the nullity invariant is introduced.
Proposition 2.4. If L is a finite semimodular lattice with height function h (h(0) = 0, h(x) = length of a maximal chain from 0 to x), then for all a, b ∈ L, h(a) + h(b) ≥ h(a ∨ b) + h(a ∧ b).
Proof. It is enough to prove the inequality in the case where a covers a ∧ b, for the general case follows by induction, adding successive covering steps and using that h(a) increases by 1 at each step. If a covers ∧ b then a ∨ b covers b (where a and b are the terms in the defining inequality of Section 2), and so h(a ∨ b) = h(b) + 1. Also h(a) = h(a ∧ b) + 1 by hypothesis. Since h(a) + h(b) = h(a ∧ b) + 1 + h(b) = h(a ∧ b) + h(a ∨ b), this base case holds with equality, the general case is then proved using the chain induction, which can only tighten the equality up to an inequality, and not vice versa.
This height based formulation of semimodularity is structurally very similar to the nullity formula of Definition 6.1: both are measuring an 'excess', the height inequality is measuring how far a ∨ b is from a + b in terms of height, the nullity formula is measuring the number of independent cycles in the cover graph that are not necessary for connectivity. It is of course no accident that Bhavale and Waphare's nullity-based enumeration programme (Section 6) is written for general dismantlable lattices, not just for semimodular ones: the adjoint-sum decomposition of Theorem 5.3 already provides, through Theorem 5.4, a similarly precise numerical understanding of the structure, without requiring the semimodular inequality at all.

3. Irreducible Elements and the Anatomy of a Finite Lattice

Definition 3.1. For a, b ∈ L, we say u covers l, written l ≺ u, if l < u and no m ∈ L satisfies l < m < u. An element a ∈ L is called meet-reducible, if there exist elements b, c ∈ L such that a = b ∧ c and b and c are both different from a, and join-reducible, if there are elements b, c ∈ L with a = b ∨ c and b and c are both different from a; a is reducible if it is meet-reducible or join-reducible, and otherwise it is called irreducible. If an element is meet-irreducible, it is also join-irreducible and vice versa. We write Red(L) and Irr(L) for the sets of reducible and doubly irreducible elements of L, respectively.
Proposition 3.2. In a finite lattice, an element has a unique lower cover iff it is join-irreducible and it has a unique upper cover iff it is meet-irreducible.
Proof. Now, let's assume that a has two different lower covers b, c. If then b ∨ c ≤ a (both below a), and b ∨ c ≠ b, c (since neither is above the other, cover a common element, and a is not above either of b, c, which would be strictly below a). If b ∨ c < a then a has no element strictly between it and each cover, b, c are the only covers considered, and one checks b ∨ c must equal a by finiteness of the interval, a = b ∨ c with b, c ≠ a, showing a join-reducible. On the other hand, if a has a unique lower cover a_*, any expression a = b ∨ c with b, c < a implies b, c ≤ a_* (because a_* is the only element immediately below a and b, c < a means b, c ≤ a_*) and hence b ∨ c ≤ a_* < a, so a cannot be equal to b ∨ c. Hence a is join-irreducible.

Dual case: meet-irreducible. The practical tool that we use throughout the paper is Proposition 3.2: If a vertex of the Hasse diagram has no lower cover except a single one, and no upper cover except a single one, then the corresponding element of L is doubly irreducible; all other vertices are reducible (apart from possibly 0 and 1, since 0 has no lower cover, while 1 has no upper cover, both of which must be checked individually):

Definition 3.3. (Rival, 1974). A finite lattice L of order n is dismantlable if there is a chain of sublattices $L_1 \subset L_2 \subset \cdots \subset L_n = L$ with $|L_i| = i$ for every i.

Proposition 3.4. With the order restricted from L, L − x is a lattice, and joins and meets in L − x are the same as those in L if x is a doubly irreducible element of L.

Proof. Let a, b ∈ L − x. As x is join-irreducible x cannot be written as p ∨ q with p, q not equal to x, so a ∨ b is not x and a ∨ b is in L − x. It is still the least upper bound of a, b in L − x, because if c is an upper bound of a, b in L − x then c is an upper bound in L and thus dominates a ∨ b. The other argument (supposing x meet-irreducible) establishes that a ∧ b ∈ L − x is also the meet computed in L – x.

The well posedness of Definition 3.3 is supported by Proposition 3.4 that deleting a doubly irreducible element always returns a lattice. The combination of Theorem 2.3 with Propositions 3.2 and 3.4 provides a comprehensive conceptual understanding: a finite lattice can be constructed order-theoretically by its join-irreducible elements and can be dismantled, combinatorially, one doubly irreducible element at a time. The theory to be surveyed in the rest of this paper is, essentially, the study of the extent and variety of different ways in which this taking apart process can continue.

4. Deletable Elements and Reducible Classes

Definition 4.1. (Bordalo–Monjardet, 1996). Consider a class $\mathcal{K}$ of finite lattices with a common structural property $\mathcal{P}$ (such as distributivity, modularity or semimodularity). If L − x is again a lattice that satisfies $\mathcal{P}$, then x is deletable. The class $\mathcal{K}$ is reducible if every L in $\mathcal{K}$, where L has more than one element, has a deletable element.

It follows from Proposition 3.4 that all doubly irreducible elements are also deletable as far as the property "is a lattice" is concerned. A substantive theoretical question is thus what happens for a stronger property $\mathcal{P}$: Does a deletable element necessarily have to have two lower covers or is it possible that an element with two lower covers or two upper covers is deletable, although its deletion does not happen to affect $\mathcal{P}$? According to Bordalo and Monjardet's analysis, the answer is a sensitive function of $\mathcal{P}$.

Theorem 4.2 (Bordalo–Monjardet, 1996). The classes of pseudocomplemented lattices, semimodular lattices and locally distributive lattices are all reducible.

The proof for each case is similar in outline and it is worth describing as an example of the way in which the forbidden-sublattice characterisations of Section 2 are used. One, where L is given and |L| > 1, presents an explicit candidate x, usually an atom, coatom or doubly irreducible element distinguished by the pseudocomplementation structure or the structures of the semimodular covering inequality, and checks that no new forbidden configuration (in this case, an N_5, for the modular-type arguments, or a violation of the semimodular covering inequality directly) is created by omitting x from L that is not already present in L. The defining inequalities of semimodularity and local distributivity are local, that is, they need only consider covering pairs, and thus the verification is confined to a finite number of covering relations involving x, and not the entire lattice.

Theorem 4.3 (Kharat–Waphare, 2001). Although the class of upper semimodular lattices is reducible, there is a finite upper semimodular poset that is not reducible.

The theoretical relevance of Theorem 4.3 lies in the fact that the lattice hypothesis is used in the proof of Theorem 4.2 just at the point where the ability of always finding a candidate deletable element is required; and that ability is lost if one assumes merely the local semimodular covering condition on a general poset. Later, Agalave, Shewale and Kharat extended the characterisation programme of Theorem 4.2 to seven more classes of lattices: chains, graded, complete, planar, algebraic, relatively atomic, and locally modular lattices and showed the reducibility of each class by the same local-verification strategy.

5. Dismantlability and the Adjunct-Sum Decomposition

Definition 3.3 is a constructive characterisation of dismantlable lattices, one deletion at a time, but provides no global characterisation of what a dismantlable lattice is like. Kelly and Rival provided the forbidden-substructure perspective.

Theorem 5.1 (Kelly–Rival, 1974). ctive. Theorem 5.1 (Kelly–Rival, 1974). A finite lattice is dismantlable if and only if it has no crown retract, i.e. a retract of the form $[m] = \{a_1, \ldots, a\_m, b_1, \ldots, b\_m\}$ with $a_i < b_i$ and $a_i < b\_\{i+1\}$ (indices mod m) and no other relation.

An immediate corollary, also due to Kelly and Rival, is that every finite planar lattice is dismantlable, because a crown cannot be drawn without edge crossing, and hence this lattice cannot be a retract of a planar lattice, when $m \geq 3$; the converse is false, however, because the Boolean lattice 2^3 is crown-free (hence dismantlable) but not planar. The result demonstrates that planarity is a strictly stronger, but sufficient, condition for dismantlability, which is of some significance in the later semimodular-lattice work on slim rectangular lattices.

The most advanced theoretical result in this domain is, the explicit structural decomposition of dismantlable lattices provided by Thakare, Pawar and Waphare using the adjoint operation.

Definition 5.2. (Adjunct sum). Define $\leq$ on $L = L_1 \cup L_2$ by keeping the order within L_1 and within L_2 unchanged, and setting $x \leq y$ whenever $x \in L_1$, $y \in L_2$ and $x \leq a$ in L_1, or $x \in L_2$, $y \in L_1$ and $b \leq y$ in L_1. Then L is again a lattice, which is called the adjunct of L_1 with L_2 with respect to the adjunct pair (a, b) and is written $L = L_1]^b_a L_2$.

Geometrically, the construction places the entirety of L_2 into the "gap" between a and b in L_1 because the containment $a \not\prec b$ ensures there is space for some element to be strictly in between a and b. This yields exactly two new covering edges $\langle a, 0\_\{L_2\}\rangle$ and $\langle 1\_\{L_2\}, b\rangle$ and no existing covering relation in L_1 or L_2 is lost.

$|E(L)| = |E(L_1)| + |E(L_2)| + 2,$

where $E(\cdot)$ denotes the set of covering edges. A lattice is an adjunct of $L_0, \ldots, L\_k$ if it has the iterated form $L = L_0]^\{b_1\}\_\{a_1\} L_1]^\{b_2\}\_\{a_2\} \cdots]^\{b\_k\}\_\{a\_k\} L\_k$.

Theorem 5.3 (Thakare–Pawar–Waphare, 2002). A finite lattice is demolishable if and only if it is an adjunct of chains.

The theoretical significance of Theorem 5.3 is that it is a passage from a purely combinatorial reduction process (Definition 3.3, removing one doubly irreducible element at each step) to an explicit algebraic decomposition (iterated adjunct of chains) in analogy with the decomposition of a finite abelian group as a direct sum of cyclic groups. The forward implication is demonstrated by extracting at each stage a maximal chain containing all the elements currently remaining that are reducible to a chain, and proving that every other element of the lattice can be attached to it as an adjunct summand, and hence is an adjunct of chains; the reverse implication is proved by noting directly from Definition 5.2 that an adjunct of chains contains no crown, so the conclusions of Theorem 5.1 apply.

Theorem 5.4 (Thakare–Pawar–Waphare, 2002). Suppose that a dismantlable lattice L has n elements and n + k edges, then $n - 2k - 2 \leq |\mathrm{Irr}(L)| \leq n - 2$.

This inequality is the theoretical connection to the refinement of the next section, which is based on the nullity of a graph: it indicates that the "excess" number of edges $n + k - n = k$ of a dismantlable lattice, in comparison to the $n - 1$ edges of a chain on n elements, has direct control over the distance of which the lattice is away from being constructed entirely out of doubly irreducible (deletable) elements.

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5.1 Planarity and the congruence-lattice perspective

A finite lattice is planar if it can be represented by a Hasse diagram that is planar in the geometric sense, that is, has no crossing edges. Planarity implies dismantlability but not vice versa as mentioned above. In the planar dismantlable lattices, a further theoretical refinement, more or less independent of the Indian enumeration tradition, has been developed by Grätzer and Czédli, based on the notion of "slim" rectangular lattice: planar semimodular lattices with exactly one doubly irreducible element in each of their left and right boundary chains, with the pair being complements of each other. By the Czédli–Schmidt sequence construction Grätzer demonstrated that for such a lattice the number of dual atoms in the congruence lattice Con(L) is exactly the number of the lengths of intervals from the two boundary doubly irreducible elements to the top element of the lattice.

This congruence-theoretic perspective is complementary, not identical, to the enumeration-theoretic perspective of Sections 6-8: both are based on the same underlying fact (Proposition 3.4), the former is about the algebra of congruences of L and the latter is about the combinatorics of counting and dimension. A structural lattice that is slim-rectangular by both analysis techniques would, in principle, be amenable to both analyses; however, there has not yet been explicit study of such lattices.

6. Nullity, Basic Blocks, and RC-Lattices

Definition 6.1. (Bhavale–Waphare, 2020). For a graph G with l vertices, m edges and n connected components, the nullity of G is $\eta(G) = m - l + n$. The nullity of a finite poset P is $\eta(P) = \eta(C(P))$, the nullity of its cover graph.

Since the cover graph of a chain on l elements is a path, with $l - 1$ edges, l vertices and 1 component, $\eta = (l - 1) - l + 1 = 0$; and conversely, one can show that $\eta(P) = 0$ forces the cover graph to be a tree with no branching, i.e. a chain. Thus the nullity 0 really describes the chains, and Theorem 5.4 is as clear directly from the total order (a meet or join of two different elements of a chain is the smaller or larger of them, and not a third element).

Definition 6.2. A finite lattice L is an RC-lattice if the elements of Red(L) are pairwise comparable, i.e. lie on a single chain of L.

Theorem 6.3 (Bhavale–Waphare, 2020). All RC-lattices are dismantlable and for any dismantlable lattice L with nullity k, the number r of reducible elements of L is |Red(L)| and satisfies

$2 \leq r \leq 2k$ *(for $k \geq 1$).*

The theoretical importance of this inequality is that nullity is not just a size but also a measure of structural complexity: the lattice might be arbitrarily large, yet have nullity 1 (or equivalently, exactly 2 reducible elements), by making the chains in its adjunct-of-chains representation longer, as lengthening a chain summand adds only doubly irreducible elements, by Proposition 3.4, and does not affect the number of reducible elements or the nullity. On the other hand, adding one nullity can increase the number of reducible elements by at most two, thus the number of reducible elements increases at most linearly with the nullity, even for large n.

Theorem 6.4 (Bhavale–Waphare, 2020). An RC-lattice of nullity l is an adjunct of l + 1 chains starting from a maximal chain with all the reducible elements if and only if L is a dismantlable lattice on n elements.

A single illustrative enumeration formula

To put this theory into a concrete result, the one closed-form enumeration theorem that will be used and verified below is recorded; further, more elaborate formulas exist in the literature at higher nullity for r = 3, 4, 5 comparable reducible elements (Bhavale–Aware, 2024, 2025; Aware–Bhavale, 2024, 2025), but the statement of the formula takes several pages of nested summation, and is not recorded here, because at this stage the emphasis is not computational, but theoretical.

Theorem 6.5 (Pawar–Waphare, 2003). The number $|\mathscr{L}(n; 2, 1)|$ of n element lattices that have exactly two reducible elements and nullity one is
$m(m - 1)(4m + 1)/6$ *if* $n = 2m + 1$, *and* $m(m - 1)(4m - 5)/6$ *if* $n = 2m$.

Example 6.6. (Verification for n = 4). Take L_1 to be the 3-chain $0 < a < 1$ and L_2 the singleton chain {y}; since $0 < 1$ in L_1 with a strictly between, (0, 1) is a valid adjunct pair, giving $L = L_1]^1_0 L_2 = \{0, a, 1, y\}$ with $0 \prec a \prec 1$, $0 \prec y \prec 1$, and a, y incomparable — the diamond lattice 2^2. Its cover graph has 4 vertices, 4 edges, 1 component, so $\eta = 4 - 4 + 1 = 1$. By proposition 3.2, the elements a and y both have a unique lower and upper cover, and therefore they are doubly irreducible; and $a \wedge y = 0$ and $a \vee y = 1$, with a, y not 0 or 1 demonstrate that 0 and 1 are not doubly irreducible. Hence $L \in \mathscr{L}(4; 2, 1)$. In the smallest non-trivial case, n = 4 (m = 2), Theorem 6.5 says that $|\ell(4; 2, 1)| = 2 \cdot 1 \cdot (8 - 5)/6 = 1$, which means that 2^2 is, up to isomorphism, the only lattice in this class, which can indeed be constructed directly. This is also an example of Theorem 6.4, which is illustrated here as an adjunct of the two chains L_1 and L_2 shown above: l + 1 = 2 chains, a result of the nullity l = 1.

7. The Reducibility Number

The reducibility defined in 4.1 is a binary property of a class: either all the lattices in the class contain a deletable element or not. Kharat, Waphare and Thakare further developed this idea to a numerical invariant of a single lattice, namely the number of successive deletions required to obtain the trivial lattice, being in the same class at each step.

Definition 7.1. (Kharat–Waphare–Thakare, 2007). The reducibility number ρ(L) of a class L is the length of the shortest sequence $L = L_n \supset L_{n-1} \supset \cdots \supset L_1$, $|L_i| = i$, that is, every L_i is obtained by removing a deletable element from L_{i+1}, and that is the shortest among all such sequences in L..

Theoretically, the values of ρ(L) always lie in the interval $[1, |L| - 1]$ and the right end $|L| - 1$ of the interval is reached exactly when L can be dismantled until a single point without leaving the area of $\mathcal{K}$ at any time in between. This is automatic by Proposition 3.4 if L is dismantlable but for a special class, say modular lattices, it is a real restriction, and it is a goal of Kharat, Waphare and Thakare's paper to prove that this maximum does exist in several classical families of lattices, or to provide sharp lower bounds when it doesn't.

Example 7.2. The sequence $L \supset \{0, y, 1\} \supset \{0, 1\} \supset \{0\}$ — where the doubly irreducible element a has been deleted from L and the doubly irreducible element y has been deleted from the resulting 3-chain, and either endpoint of the resulting 2-chain has been deleted — is a valid dismantling sequence for the diamond lattice L of Example 6.6, and thus ρ(L) = 3, the maximum possible for a 4-element lattice.

8. The Order Dimension of RC-Lattices

Definition 8.1. (Dushnik–Miller). A realizer of a finite poset P is a family of linear extensions of P whose intersection (as relations) is the order of P. The order dimension dim(P) is the smallest possible size of a realizer.

Order dimension is a similarity measure for posets, different from nullity, to the extent that a chain has dimension 1 (as there is only one linear extension that realizes it), and a general finite poset may require arbitrarily many linear extensions to reconstruct its order from the intersection of total orders. Thus it is a priori surprising that RC-lattices with arbitrarily large nullity and number of reducible elements can be constructed by the constructions of Section 6, and that nevertheless the dimension of these lattices is bounded.

Theorem 8.2 (Bhavale, 2025). All RC-lattices have order dimension 3 at most; this upper bound is attained.

The idea behind the proof is to use the above Theorem 6.4 as an adjunct of chains representation: if dim L_2 is small enough, then only a few additional linear extensions in L_2 are needed to place it by the interval (a, b) and not by L_1. Since a chain trivially contributes dimension 1 when it stands on its own, and the introduction of the adjunct operation is seen to contribute at most a bounded increment to the dimension of the chain it is attached to, an induction over the $l + 1$ chains of an RC-lattice's adjunct representation gives a bound that is independent of l — that is, independent of both n and the nullity k.

Example 8.3. There is only one covering relation for the diamond lattice 22 of Example 6.6, the two linear extensions $0 < a < y < 1$ and $0 < y < a < 1$, and dim(22) = 2 is realized by these two extensions: their intersection is exactly the covering relations of 22, and they order a, y oppositely, while every other pair of elements is ordered the same way by both of them. This is still well within the bound dim ≤ 3 of Theorem 8.2, which is only achieved by larger basic blocks, but not by 2^2 itself.

In a sense, Theorem 8.2 is the theoretical counterpart of the enumeration formulas of Section 6: the class of RC-lattices, combinatorially speaking, is unbounded and its enumeration formulas become increasingly complicated (depending on the number of reducible elements), but on the other hand, the whole class is very simple from the point of view of the single invariant dim.

9. Application: Frankl's Union-Closed Sets Conjecture

In every finite union-closed family of sets (excluding the trivial family {∅}) there is an element in at least half of the members of the family. The conjecture can be expressed as a purely lattice-theoretic statement, via the standard correspondence between a union-closed family and the finite lattice it generates under the union closure (as join) and the meet defined in the obvious way.

Lattice form of the conjecture. For any finite lattice L ($|L| > 1$), there exists some join-irreducible j with $|{\downarrow}j| \leq |L|/2$, where ${\downarrow}j = \{x \in L : x \leq j\}$.

Theorem 9.1 (Joshi–Waphare–Kavishwar, 2016). Frankl's conjecture, that the lattice form is true for dismantlable lattices, is correct.

The proof is by induction on the number of chains in the adjunct-of-chains representation given in Theorem 5.3. In this particular base case, L is one chain with its minimum non-zero element having the principal ideal of size 1, which is trivially $\leq |L|/2$ for $|L| \geq 2$. In the inductive step, writing $L = L_0 \,]^{b}_{a}\, L_1 \cdots$, the join-irreducible element guaranteed to be in L_0 by the induction hypothesis is shown, using the edge-count identity of Section 5, to continue to have the required bound in the larger lattice L, since adjoining L_1 only adds elements above b or from the interval (a, b), and not enlarge the principal ideal of any join-irreducible element that already lay strictly below a in L_0. The theoretical interest of this result, from the point of view of this paper, is that it provides substantive explanatory power for an adjunct-sum machinery which was introduced solely for the purpose of describing and counting dismantlable lattices, for an intractable, yet still partially open, problem in extremal combinatorics.

10. Discussion and Conclusion

The surveyed theory in this paper has a coherent logical progression. It is shown that a finite lattice is determined by its join-irreducible elements (Theorem 2.3), that join-irreducible elements are precisely those with a unique lower cover (Proposition 3.2), that the doubly irreducible elements (those that are simultaneous simplest from the join and the meet side) can always be deleted (Proposition 3.4), that the same underlying representation gives rise to a numerical refinement in terms of nullity (Theorem 6.3), which entails both the enumeration result of Section 6 and the dimension bound of Section 8. In section 9, the application to Frankl's conjecture illustrates that this strand of thinking, which has its origins in classification, has definite scope beyond enumeration.

The following table summarizes for the RC-lattice case, which combinations of the number of reducible elements r and the nullity k are completely solved by an explicit enumeration theorem in the literature reviewed above.

(r, k)	Status	Source
(0, 0)	Resolved — unique lattice for each n is the chain	Proposition, this paper (§6)
(2, 1)	Resolved — closed form in n	Pawar–Waphare, 2003
(2, k), k ≥ 2 and (3, k), k ≥ 2	Resolved	Bhavale–Aware, 2024–2025
(4, 2), (4, 3), (5, 3)	Resolved	Bhavale–Aware / Aware–Bhavale, 2024–2025
r ≥ 6 or k ≥ 4	Open in general	—
non-comparable Red(L)	Open outside the RC-lattice case	—

There are a number of theoretical questions which are still open. First, if there is a common mechanism (plausibly involving the forbidden N_5/M_3 configurations of Theorems 2.1–2.2) that makes a class of lattices deletable, then it must be identified, but no uniform characterisation of deletability of all classes of lattices has been given so far. Second, Theorem 6.3 and the enumeration and dimension results are valid only for RC-lattices where the reducible elements are all pairwise comparable; it is known that for r ≥ 4, this does not have to be the case but no structural decomposition similar to that of Theorem 6.4 has been provided for the general case. Thirdly, it would be interesting to find out if the bound of Theorem 8.2, perhaps with a larger but still absolute constant, also holds true for dismantlable lattices in general, and not just for RC-lattice as is case in Bhavale's proof. To sum up: This paper has contained a more or less self-contained theoretical treatment, based on a really small number of formulas, each exactly stated and, where possible, backed by explicit construction, of the structure theory of deletable elements in finite lattices, from the basic representation and dismantlability theorems to the adjoint-sum decomposition and its numerical (nullity, reducibility number, dimension) and combinatorial (Frankl's conjecture) consequences, and has pointed out the main directions in which the theory is not yet complete.

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